PMM U.S.S.R., Vol.50, No.4, pp.507-515, 1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

CONTACT BETWEEN AN ELASTIC HALF-PLANE AND A PARTLY SEPARATED STAMP*

E.L. NAKHMEIN and B.M. NULLER

Problems for an elastic half-plane in contact with a stamp (or stamps) partly with full coupling, partly in contact with slippage, are considered. The problems reduce to the combined Dirichlet-Riemann boundary value problem and can, in general, be solved in quadratures. Solutions of homogeneous problems are given in terms of elementary functions. All combinations of the geometrical, force and other parameters at which contact with slippage occurs in a single region are shown, formulas showing the extent of the zones of smooth adhesion of the separated layers of the contacting bodies are given, and stress intensity coefficients are found for the cracks where separation into layers occurs and at the stamp edges.

Such problems were first studied by Galin /2/ and Fal'kovich /3/ who assumed that the stamps are pressed into the half-plane. Later considerable attention was given to the problems of the detachment of the separated layers and the mathematically equivalent problems of crack propagation at the boundary separating the materials /4/. The main problem encountered when studying the strength characteristics of the elastic regions indicated above within the region of stress concentration is already clearly seen in the solution of the problem due to Abramov /5, 6/, and is connected with the fact that the above solution cannot be used in the case of the separation of a stamp partly detached from the support. The translational oscillations of the free boundary of the half-plane at the edge of the zone of full coupling mean in fact that the half-plane and the stamp intersect. The reliable realization of numerical, approximate and asymptotic solutions of problems of this type is hindered by their instability, as is seen from the results given below.

1. Let the boundary of the elastic half-plane $-\infty < x < \infty$, $y \leqslant 0$ be in contact with the coupled and a slipping stamp on the segments $x \in [a, b] = M$ and $x \in [c, d] = L$, $a < b < c < d$, so that the following conditions hold:

$$u(x) = u_0(x) + r_0,\ x \in M;\quad v(x) = v_0(x) + r(x),\ x \in L \cup M \tag{1.1}$$
$$\tau_{xy}(x) = 0,\quad x \in L;\quad \sigma_y(x) = \tau_{xy}(x) = 0,\quad x \in L'$$

Here $u_0(x)$ and $v_0(x)$ are real functions defining the tangential tension and the form of the stamps and their derivatives satisfy the Hölder condition, $r(x) = r_1$ on M, $r(x) = r_2$ on L; r_0, r_1, r_2 are real constants, L' is the complement of $L \cup M$ to the real axis. Using the given load acting on the stamps and the condition at infinity, it is required to find the stress-strain state of the half-plane.

We shall seek the solution of problem (1.1) in the form due to Muskhelishvili /6/

$$\sigma_y - i\tau_{xy} = \Phi(z) - \Phi(\bar z) + (z - \bar z)\overline{\Phi'(z)} \tag{1.2}$$
$$2\mu(u' + iv') = \varkappa\Phi(z) + \Phi(\bar z) - (z - \bar z)\overline{\Phi'(z)}$$

$$\Phi(z) = {}^1/_4\sigma_x^\infty + 2i\mu(\varkappa + 1)^{-1}\varepsilon^\infty - Fe^{i\theta}(2\pi z)^{-1} + O(z^{-2}),\quad z \to \infty \tag{1.3}$$

where the function $\Phi(z)$ is analytic in the plane of the complex variable $z = x + iy$ with cuts along L and M, and has no more than integrable singularities at the nodal points; a prime denotes a derivative with respect to x; $\varkappa = 3 - 4\nu$, ν is Poisson's ratio, μ is the shear modulus, σ_x^∞ is the stress, ε^∞ is the rotation at infinity, and F and θ are the magnitude and the angle of inclination of the principal vector of the forces applied to the stamps; the angle θ is measured anticlockwise from the direction of the axis Ox, $0 \leqslant \theta < 2\pi$.

Substituting (1.2) into (1.1), we obtain for $\Phi(z)$ the combined Dirichlet-Riemann boundary value problem /1/

$$\operatorname{Im}\Phi^\pm(x) = f(x),\quad f(x) = 2\mu(\varkappa + 1)^{-1}u_0'(x),\quad x \in L \tag{1.4}$$

*Prikl.Matem.Mekhan., 50, 4, 663-673, 1986

$$\Phi^+(x) + \varkappa\Phi^-(x) = g(x), \quad g(x) = 2\mu\,[u_0'(x) + iv_0'(x)] \tag{1.5}$$
$$x \in M$$

We will construct the canonical solution $X(z)$ of the homogeneous boundary value problem

$$\operatorname{Im}\Phi^\pm(x) = 0, \quad x \in L \tag{1.6}$$

$$\Phi^+(x) + \varkappa\Phi^-(x) = 0, \; x \in M \tag{1.7}$$

in the form /1/ which will immediately satisfy condition (1.7). We obtain

$$X(z) = Z(z)\, e^{i\psi(z)} (z-d)^{-\alpha} \tag{1.8}$$

$$Z(z) = (z-a)^{-1/2+i\gamma}(z-b)^{-1/2-i\gamma}, \quad \gamma = 1/2\pi^{-1}\ln\varkappa \tag{1.9}$$
$$0 \leqslant \arg(z-a) \leqslant 2\pi, \; 0 \leqslant \arg(z-b) \leqslant 2\pi$$

Here $Z(z)$ is a canonical solution of the homogeneous Riemann problem (1.7) with integrable singularities at the nodes: the analytic function $\psi(z)$ is bounded in the z plane, with a cut along L, and has no more than logarithmic singularities at the ends of the cut L, and α is an integer.

Condition (1.6) is equivalent to the equations

$$\arg X^\pm(x) = \pi n^\pm, \; x \in L \tag{1.10}$$

where $n^\pm$ are integers. Substituting (1.8) into (1.10) we obtain the following Dirichlet problem for determining the function $\psi(z)$:

$$\operatorname{Re}\psi^\pm(x) = h^\pm(x), \quad x \in L \tag{1.11}$$
$$h^\pm(x) = \pi n^\pm - \arg Z^\pm(x) + \alpha \arg(x-d)^\pm$$

whose solution in the class of functions shown above is given by the formulas /7/

$$\psi(z) = \frac{Y(z)}{2\pi i}\int_L \frac{h^+(t)+h^-(t)}{Y^+(t)(t-z)}\,dt + \frac{1}{2\pi i}\int_L \frac{h^+(t)-h^-(t)}{t-z}\,dt \tag{1.12}$$
$$Y(z) = \sqrt{(z-c)(z-d)}, \quad 0 \leqslant \arg(z-c) \leqslant 2\pi$$
$$0 \leqslant \arg(z-d) \leqslant 2\pi$$

Transforming (1.12) with help of the relations /6/

$$\int_L \frac{dt}{Y^+(t)(t-z)} = \frac{\pi i}{Y(z)} \tag{1.13}$$

and the formulas (1.11), (1.9), we obtain

$$\psi(z) = \frac{\pi}{2}(n^+ + n^- + 2) + \pi\alpha + \Psi(z) + \frac{n^+ - n^- - 2}{2i}\ln\frac{z-d}{z-c} \tag{1.14}$$

$$\Psi(z) = \frac{\gamma Y(z)}{\pi i} I(z), \quad I(z) = \int_L \frac{\ln[(t-b)(t-a)^{-1}]\,dt}{Y^+(t)(t-z)} \tag{1.15}$$

Let us pass in the expression for $I(z)$ to the double integral

$$I(z) = \int_L \frac{dt}{Y^+(t)(t-z)}\int_M \frac{d\tau}{\tau - t}$$

and following /7, Sect.28/, change the order of integration. By virtue of (1.13) we obtain ($\tau \in M$)

$$I(z) = \frac{\pi i}{Y(z)}\ln\frac{z-b}{z-a} - \pi i\int_M \frac{d\tau}{Y(\tau)(\tau - z)}$$
$$Y(\tau) = -\sqrt{(c-\tau)(d-\tau)}$$

This, together with (1.15), yields

$$\Psi(z) = \gamma\ln\frac{z-b}{z-a} + \varphi(z) \tag{1.16}$$

$$\varphi(z) = -\gamma Y(z)\int_M \frac{d\tau}{Y(\tau)(\tau-z)} \tag{1.17}$$

Evaluating the last integral we obtain

$$\varphi(z) = 2\gamma\ln\left[\sqrt{\frac{z-a}{z-b}}\,\frac{\sqrt{(d-b)(z-c)} + \sqrt{(c-b)(z-d)}}{\sqrt{(d-a)(z-c)} + \sqrt{(c-a)(z-d)}}\right] \tag{1.18}$$

$$\varphi(z)=\beta_0+\beta_1 z^{-1}+O(z^{-2}),\quad z\to\infty \tag{1.19}$$
$$\beta_0=2\gamma\ln\frac{\sqrt{d-b}+\sqrt{c-b}}{\sqrt{d-a}+\sqrt{c-a}},\quad \beta_1=\gamma\left[\sqrt{(c-a)(d-a)}-\sqrt{(c-b)(d-b)}\right]$$

It can be confirmed that $\beta_0<0,\ \beta_1>0$.

Thus the canonical solution of problem (1.6), (1.7) has, in accordance with (1.8), (1.9), (1.14), (1.16), the form

$$X(z)=-e^{1/2 i\pi(n^++n^-)}e^{i\pi\alpha}\times(z-a)^{-1/2}(z-b)^{-1/2}(z-c)^{-1/2 n+1}(z-d)^{1/2 n-1-\alpha}e^{i\varphi(z)} \tag{1.20}$$

where $n=n^+-n^-$ and the function $\varphi(z)$ is given by formula (1.18).

We find the integers $n^\pm$ and α from the behaviour of the solution at the nodes of the line L. According to (1.20), (1.18), two linearly independent canonical solutions exist in the igven class of functions, and each of these solutions has oscillating singularities at $z=a$ and $z=b$.

The first solution

$$X_1(z)=\frac{ie^{i\varphi(z)}}{\sqrt{(z-a)(z-b)(z-c)(z-d)}} \tag{1.21}$$

with root-type singularities at $z=c,\ z=d$ corresponds to the case $n^+=0,\ n^-=-3,\ \alpha=1$. The second solution

$$X_2(z)=\frac{e^{i\varphi(z)}}{\sqrt{(z-a)(z-b)}} \tag{1.22}$$

is bounded and different from zero at the points $z=c,\ z=d$, and is generated by the parameters $n^+=0,\ n^-=-2,\ \alpha=0$.

When $z\to\infty$, we obtain from (1.19), (1.21), (1.22)

$$X_1(z)=\frac{ie^{i\beta_0}}{z^2}\left[1+\frac{i\beta_1+1/2(a+b+c+d)}{z}\right]+O\left(\frac{1}{z^4}\right) \tag{1.23}$$
$$X_2(z)=\frac{e^{i\beta_0}}{z}\left[1+\frac{i\beta_1+1/2(a+b)}{z}\right]+O\left(\frac{1}{z^3}\right)$$

The general solution of the combined boundary value problem (1.4), (1.5) bounded at infinity, has the form

$$\Phi(z)=P(z)X_1(z)+Q(z)X_2(z)+\Phi_0(z) \tag{1.24}$$
$$P(z)=C_1z^2+C_2z+C_3,\ Q(z)=D_1z+D_2$$

where $P(z)$ and $Q(z)$ are polynomials with real coefficients, and $\Phi_0(z)$ is a particular solution of the inhomogeneous problem determined by the formulas /1/

$$\Phi_0(z)=X_1(z)[\Phi_1(z)+\Phi_2(z)],\ \Phi_0(z)=O(z^{-2}),\quad z\to\infty \tag{1.25}$$
$$\Phi_1(z)=\frac{1}{2\pi i}\int_M\frac{g(t)\,dt}{X_1^+(t)(t-z)},\quad \Phi_2(z)=\frac{Y(z)}{2\pi}\int_L\frac{q^+(t)+q^-(t)}{Y^+(t)(t-z)}dt+$$
$$\frac{1}{2\pi}\int_L\frac{q^+(t)-q^-(t)}{t-z}dt,\quad q^\pm(t)=\frac{f^\pm(t)}{X_1^\pm(t)}-\operatorname{Im}\Phi_1^\pm(t)$$

Let us write the formulas for the normal stresses in the segment of contact L. From (1.24), (1.21), (1.22), (1.18) and (1.3) we have

$$\sigma_y(x)=-\frac{2P(x)\operatorname{ch}\varphi_0(x)}{\sqrt{(x-a)(x-b)(x-c)(d-x)}}-\frac{2Q(x)\operatorname{sh}\varphi_0(x)}{\sqrt{(x-a)(x-b)}}-\Phi_0^+(x)+\Phi_0^-(x) \tag{1.26}$$
$$\varphi_0(x)=\gamma\left[\operatorname{arctg}\frac{\sqrt{(c-a)(d-x)}}{\sqrt{(d-a)(x-c)}}-\operatorname{arctg}\sqrt{\frac{(c-b)(d-x)}{(d-b)(c-x)}}\right]$$

where we obviously have $\varphi_0(x)\geqslant 0$ when $x\in L$.

Using the same relations, we write the expression for the derivative of vertical displacements of the boundary of the half-plane between the stamps $(b<x<c)$, in the form

$$2\mu v'(x)=(\varkappa+1)\left[-\frac{P(x)\cos\varphi_1(x)}{\sqrt{(x-a)(x-b)(x-c)(x-d)}}+\right. \tag{1.27}$$
$$\left.\frac{Q(x)\sin\varphi_1(x)}{\sqrt{(x-a)(x-b)}}\right]+\operatorname{Im}[\varkappa\Phi_0^-(x)+\Phi_0^+(x)]$$
$$\varphi_1(x)=2\gamma\ln\left[\sqrt{\frac{x-a}{x-b}}\,\frac{\sqrt{(d-b)(c-x)}+\sqrt{(c-b)(d-x)}}{\sqrt{(d-a)(c-x)}+\sqrt{(c-a)(d-x)}}\right]$$

Let us consider two versions of the formulation of the problem: a) the stamps are rigidly coupled to each other and the total principal vector of external forces applied to the stamps is given (Fig.1(1), the double line indicates sliding contact); b) the stamps move independently without rotation, under the action of known external forces applied to each stamp (Fig. 1(2)).

In both problems the constants r_0 and r_1 characterizing the displacement of the half-plane as a whole, remain undetermined; the vertical displacements are connected by the condition of continuity

$$v_0(c) + r_2 = \int_b^c v'(x)\,dx + v_0(b) + r_1 \tag{1.28}$$

The constants C_1, C_2, D_1, D_2 are found from the conditions at infinity (1.3) which generate, in accordance with (1.25), the following system of algebraic equations:

$$\begin{aligned} D_1 + iC_1 &= e^{-i\beta_0}[{}^1/_4\sigma_x^\infty + 2i\mu(\varkappa+1)^{-1}\varepsilon^\infty] \\ D_2 + iC_2 + iC_1[i\beta_1 + {}^1/_2(a+b+c+d)] + D_1[i\beta_1 + {}^1/_2(a+b)] &= -{}^1/_2\pi^{-1}Fe^{i(\theta-\beta_0)} \end{aligned} \tag{1.29}$$

Let us find C_3 and r_2. Putting in problem a) $r_1 = r_2$, we find the constant C_3 from Eq. (1.2), taking into account (1.27), (1.29).

In problem b) we assume that the normal force Y_2 applied to the stamp L is given, and obtain the following equation for determining C_3:

$$\int_L \sigma_y(x)\,dx = Y_2 \tag{1.30}$$

which should be solved together with (1.26) and (1.29); r_2 is found from (1.28).

The soluiton constructed can be realized mechanically, provided that $\sigma_y(x) \leqslant 0,\ x \in L$, which imposes constraints on the geometrical and force parameters of the problem.

2. Let us consider the case when the ends of L and M coincide. Suppose that in conditions (1.1) $M = [a, b],\ L = [b, d],\ r_1 = r_2$, which corresponds to a single stamp whose edge has separated from the support, in contact with the half-plane. It is obvious (Fig.1, (3), (4)) that the edge $x = d_0$ of the stamp which has separated from the support need not coincide with the end of the slippage segment $x = d\ (d_0 \geqslant d)$. Putting $c = b$ in (1.20) and (1.18) we obtain the general expression for the canonical solution

$$X(z) = -e^{{}^1/_2 i\pi(n^+ + n^-)}\,e^{i\pi\alpha}\,(z-a)^{-{}^1/_2}\,(z-b)^{{}^1/_2 - {}^1/_2 n}\,(z-d)^{{}^1/_2 n - 1 - \alpha}\,e^{i\varphi(z)} \tag{2.1}$$

$$\varphi(z) = 2\gamma \ln \frac{\sqrt{(d-b)(z-a)}}{\sqrt{(d-a)(z-b)} + \sqrt{(b-a)(z-d)}} \tag{2.2}$$

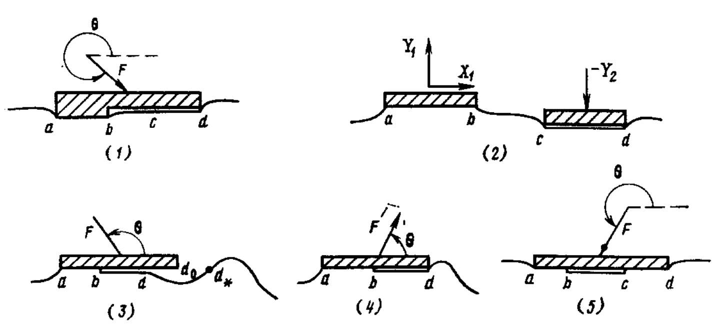

Fig.1

Since according to (2.2), $\varphi^{\pm}(b) = \mp\pi i\gamma$, it follows from (2.1) that the function $X(z)$ does not oscillate near the point $z = b$.

Just as in Sect.1, we have here two linearly independent canonical solutions with oscillating singularities at $z = a$. One of them, with a root-type singularity at the point $z = d$,

$$X_1(z) = \frac{ie^{i\varphi(z)}}{\sqrt{(z-a)(z-d)}} \tag{2.3}$$

is bounded and non-zero at $z = b$. It corresponds to the parameters $n^+ = 0,\ n^- = -1,\ \alpha = 0$. The other solution

$$X_2(z)=\frac{e^{i\varphi(z)}}{\sqrt{(z-a)(z-b)}} \tag{2.4}$$

is generated by the parameters $n^+=0, n^-=-2$, $\alpha=0$, bounded at $z=d$, and has an integrable singularity at $z=b$.

The asymptotic forms of these solutions at infinity are given by the formulas

$$X_1(z)=\frac{ie^{i\beta_0}}{z}\left[1+\frac{i\beta_1+{}^1/_2(a+d)}{z}\right]+O\left(\frac{1}{z^3}\right) \tag{2.5}$$
$$X_2(z)=\frac{e^{i\beta_0}}{z}\left[1+\frac{i\beta_1+{}^1/_2(a+b)}{z}\right]+O\left(\frac{1}{z^3}\right)$$

where, according to (1.19),

$$\beta_0=2\gamma\ln\frac{\sqrt{d-b}}{\sqrt{d-a}+\sqrt{b-a}}<0,\quad \beta_1=\gamma\sqrt{(b-a)(d-a)} \tag{2.6}$$

The general solution of the problem, bounded at infinity, is given by formula (1.24) where

$$P(z)=C_1z+C_2,\quad Q(z)=D_1z+D_2 \tag{2.7}$$

The asymptotic form of the particular solution of the inhomogeneous problem (1.25) is, as $z\to\infty$,

$$\Phi_0(z)=-\frac{q_0e^{i\beta_0}}{z}+O\left(\frac{1}{z^2}\right),\quad q_0=\frac{1}{2\pi}\int\limits_L\frac{q^+(t)+q^-(t)}{\sqrt{(t-b)(d-t)}}\,dt \tag{2.8}$$

Let us write the formulas for the contact stresses in the slippage segment L:

$$\sigma_y(x)=-\frac{2P(x)\operatorname{ch}\varphi_0(x)}{\sqrt{(x-a)(d-x)}}-\frac{2Q(x)\operatorname{sh}\varphi_0(x)}{\sqrt{(x-a)(x-b)}}-\Phi_0^+(x)+\Phi_0^-(x) \tag{2.9}$$
$$\varphi_0(x)=\gamma\operatorname{arctg}\sqrt{\frac{(b-a)(d-x)}{(d-a)(x-b)}}$$

and in the coupled segment M:

$$(\sigma_y-i\tau_{xy})(x)=\frac{\varkappa+1}{\sqrt{\varkappa}}\left[\frac{P(x)}{\sqrt{(x-a)(d-x)}}-\frac{iQ(x)}{\sqrt{(x-a)(d-x)}}\right]e^{i\varphi_0(x)}- \tag{2.10}$$
$$\Phi_0^+(x)+\Phi_0^-(x),\quad \varphi_0(x)=2\gamma\ln\frac{\sqrt{(d-b)(x-a)}}{\sqrt{(d-a)(b-x)}+\sqrt{(b-a)(d-x)}}$$

We find the arbitrary constants from the conditions at infinity. As in (1.29), we have

$$D_1+iC_1=[{}^1/_4\sigma_x^\infty+2i\mu(\varkappa+1)^{-1}\varepsilon^\infty]e^{-i\beta_0} \tag{2.11}$$
$$D_2+iC_2+iC_1[i\beta_1+{}^1/_2(a+d)]+D_1[i\beta_1+{}^1/_2(a+b)]=$$
$$-{}^1/_2\pi^{-1}Fe^{i(\theta-\beta_0)}+q_0$$

Let us consider the case of a flat stamp $(u_0(x)=v_0(x)\equiv0)$ in translational motion caused by the action of a force F, which makes an angle θ with the Ox axis. Let $\sigma_x^\infty=\varepsilon^\infty=0$, then $\Phi_0(z)\equiv0$ and from (2.11) we have

$$C_1=D_1=0,\quad C_2=-{}^1/_2\pi^{-1}F\sin(\theta-\beta_0),\quad D_2=-{}^1/_2\pi^{-1}F\cos(\theta-\beta_0) \tag{2.12}$$

According to (2.9), (2.10), (2.12) the asymptotic form of the stresses at the points where the boundary conditions separate, is

$$\sigma_y(x)=\sigma_0+O(\sqrt{b-x}),\quad \tau_{xy}(x)=K_{\mathrm{II}}[2\pi(b-x)]^{-1/2}+ \tag{2.13}$$
$$O(\sqrt{b-x}),\quad x\to b-0$$
$$\sigma_y(x)=K_{\mathrm{I}b}[2\pi(x-b)]^{-1/2}+\sigma_0+O(\sqrt{x-b}),\quad x\to b+0$$
$$\sigma_y(x)=K_{\mathrm{I}d}[2\pi(d-x)]^{-1/2}+O(\sqrt{d-x}),\quad x\to d-0$$

$$K_{\mathrm{I}b}=(\varkappa-1)(\varkappa+1)^{-1}K_{\mathrm{II}}, \tag{2.14}$$
$$K_{\mathrm{II}}=(\varkappa+1)F\cos(\theta-\beta_0)[2\pi\varkappa(b-a)]^{-1/2}$$

$$K_{\mathrm{I}d}=\sqrt{2}F\sin(\theta-\beta_0)[\pi(b-a)]^{-1/2} \tag{2.15}$$
$$\sigma_0={}^1/_2\pi^{-1}(\varkappa+1)F\sin(\theta-\beta_0)[\varkappa(b-a)(d-b)]^{-1/2}$$

Using formulas (2.9), (2.12), we can confirm that the contact stresses will be compressive over the whole slippage segment L, provided that

$$\sin(\theta-\beta_0)\leqslant 0,\quad \cos(\theta-\beta_0)\leqslant 0 \tag{2.16}$$

Hence, taking into account the fact that $\beta_0<0$, we obtain from this the necessary and sufficient conditions for the solution to be mechanically meaningful, in the form of a system of inequalities:

$$\pi(2k+1)\leqslant\theta-\beta_0\leqslant\pi(2k+{}^3/_2)\quad k=0,1,2,\ldots \tag{2.17}$$

If the lengths of the segments with coupling and with slippage are given, then β_0 takes a specific value (2.6) and relations (2.17) will restrict the direction of the force which produces a contact along the whole segment L. If the direction of theforce is given, then the inequalities can be written in a different form, establishing the dimensions of the slippage zone L. Let us write $\lambda=(d-b)(d-a)^{-1}$ and $\lambda_0=(d_0-b)(d_0-a)^{-1}$ for the relative length of the slippage and separation zones. Transforming (2.6) to the form $\beta_0=\gamma\ln[(1-\sqrt{1-\lambda})(1+\sqrt{1-\lambda})^{-1}]$ and substituting it into (2.17), we obtain

$$\begin{aligned}&m_k(\theta)\leqslant\lambda\leqslant p_k(\theta),\quad k=0,1,2,\ldots\\&p_k(\theta)=[\operatorname{ch}W_k(\theta)]^{-2},\quad m_k(\theta)=p_k(\theta-{}^1/_2\pi)\\&W_k(\theta)=\max\{0,{}^1/_2\gamma^{-1}[\pi(2k+1)-\theta]\}\end{aligned} \tag{2.18}$$

Thus a solution containing one coupled segment and one slippage segment will be realized in any case, provided that $d_0=d$ (Fig.1(4)). Moreover, when the direction of the force is given, the relative length λ of the slippage segment will be found in the denumerable set of segments (2.18), which become narrower as k increases.

When $m_k(\theta)\leqslant\lambda<p_k(\theta)$, the condition $d_0=d$ is necessary since according to (2.15), (2.14) and (2.16) we find, that near the point $x=d$ $v'(x)>0$ and at any $d_0>d$, the free boundary of the half-plane would intersect the stamp.

If on the other hand

$$\lambda=p_k(\theta) \tag{2.19}$$

then the admissible length of the cut will always be greater than the length of the zone of contact $(d_0>d)$, and the intervals of existence of a mechanically realizable solution with a single slippage segment (2.18) can therefore be extended (Fig.1(3)). Indeed, in this case we have

$$\beta_0=\beta_{k1}(\theta),\ \beta_{k1}(\theta)=\theta-\pi(2k+1),\ C_2=0,\ D_2={}^1/_2\pi^{-1}F \tag{2.20}$$

and by virtue of (2.13), (2.14)

$$\begin{aligned}&\sigma_y(x)=-\frac{F(\varkappa-1)}{2\pi\sqrt{\varkappa(b-a)(x-b)}}+O(\sqrt{x-b}),\quad x\to b+0\\&\sigma_y(b-0)=0\\&\tau_{xy}(x)=-\frac{F(\varkappa+1)}{2\pi\sqrt{\varkappa(b-a)(b-x)}}+O(\sqrt{b-x}),\quad x\to b-0,\quad \sigma_y(d)=0\end{aligned} \tag{2.21}$$

Using (1.2), (1.24), (2.2)-(2.4), (2.12) and (2.20), we obtain

$$\begin{aligned}&2\mu(\varkappa+1)^{-1}v'(x)=\frac{F\sin\varphi(x)}{\sqrt{(x-a)(x-b)}},\quad x\geqslant d\\&\varphi(x)=2\gamma\ln\frac{\sqrt{(d-b)(x-a)}}{\sqrt{(d-a)(x-b)}+\sqrt{(b-a)(x-d)}}\end{aligned} \tag{2.22}$$

From this it follows that when $x>d$ $\varphi'(x)<0$, the function $\varphi(x)$ decreases monotonically and

$$\beta_{k1}(\theta)<\varphi(x)\leqslant 0,\ x\geqslant d \tag{2.23}$$

From (2.22), (2.23) we find $v'(d+0)=0$, which means that at the point $x=d$ and under conditions (2.19), the surfaces of the stamp and the half-plane which are in contact with each other, adhere smoothly to one another and $v(x)<v(d)$ in a certain neighbourhood of this point. The free boundaries of the half-plane intersects an extension of the stamp surface at certain points $x=x_*$, provided that $v(x_*)=v(d)$. By virtue of (2.22), the latter condition is equivalent to the following equation in x_*:

$$\int_d^{x_*}\frac{\sin\varphi(x)\,dx}{\sqrt{(x-a)(x-b)}}=0 \tag{2.24}$$

in which the value of d is found from (2.19), depending on the value of k.

This implies that $d_0\in(d,d_*)$ where $d_*=\min x_*$.

The relations (2.23) (2.20) when $k=0$ $0\leqslant\theta\leqslant\pi$, i.e. when $-\pi\leqslant\beta_{k1}(\theta)\leqslant 0$, show that Eq.(2.24) has no solutions. Thus, if the force pulls the stamp away from the half-plane $(0\leqslant\theta\leqslant\pi)$, then $d_*=\infty$.

The table below gives the values of the relative length of the slippage zone (2.19) when

$k = 0$, depending on the direction of the force and the magnitude of Poisson's ratio.

Table

$\varkappa$	$\theta=0$	$\pi/4$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	$7\pi/8$	π
3	$6{,}29\cdot10^{-8}$	$5{,}62\cdot10^{-6}$	$5{,}02\cdot10^{-4}$	$9{,}97\cdot10^{-3}$	$4{,}38\cdot10^{-2}$	0,182	0,346	1
1,8	$1{,}04\cdot10^{-14}$	$4{,}61\cdot10^{-11}$	$2{,}04\cdot10^{-7}$	$5{,}50\cdot10^{-5}$	$9{,}03\cdot10^{-4}$	0,015	0,058	1

When $k \geqslant 1$, the quantity d_* will always be finite. This follows from the theorem on the uniqueness of the solution of contact problems in the theory of elasticity containing a region of one-sided constraints, which can be proved by following /8/. Thus the possible length of the cut will satisfy, not (2.18) but the extended inequalities

$$m_k(\theta) \leqslant \lambda_0 < \lambda_*, \quad \lambda_* = (d_* - b)(d_* - a)^{-1} \tag{2.25}$$

In all the remaining cases $\lambda_* \leqslant \lambda_0 < m_{k-1}(\theta)$, contact between the separated part of the stamp and the half-plane occurs along two or more separate segments.

In conclusion, we will trace the development of the separation crack.

From (2.14) it follows that when conditions (2.16) hold, the stress intensity coefficient σ_y at the point $x = b$ is always non-positive. Therefore, the growth of the closed crack L (separation) can be caused, from the point of view of the theory of the fracture of elastic materials /9, 10/, only by the tangential stresses whose singular character near the point $x = b$ has the form (2.13), (2.14). Since the function $\tau_{xy}(x)$ is continuous in β_0 and $\tau_{xy}(b) = 0$ when $\beta_0 = \theta - \pi(2k + {}^3/_2)$, we can have as many segments of the stamp foundation as we like, determined by the number k on the left-hand side of inequality (2.18) which, when occupied by the separation crack, leave the latter stable, according to the theory of cracks, for any material and any value of F. (We note that the stresses $\sigma_y(b)$ are bounded, but very large; if $\sigma_* = -(d-a)F^{-1}\sigma_y(b)$ and $\varkappa = 3$, then when $k = 0$, $\theta = {}^1/_2\pi$ $\sigma_* = 1.46\cdot10^3$, when $k = 1$, $\theta = {}^3/_2\pi$ $\sigma_* = 1.17\cdot10^7$, when $\varkappa = 1.8$, $k = 0$, $\theta = {}^1/_2\pi$ $\sigma_* = 3.25\cdot10^6$.) The growth of the crack on the segments indicated can be caused only by the non-elastic factors such as plasticity, temperature or chemical reaction, and will, naturally, proceed slowly or will stop completely.

However, if for any $0 < \theta < \pi$ the crack should shift, for whatever reason, or a notch is made extending to the point corresponding to Eq.(2.19) at $k = 0$, then the crack will become globally unstable, or in the course of its growth smooth contact will be maintained at the point $x = b$, with one segment of contact of L retained and the tangential stress intensity coefficient $K_{II} = -(\varkappa + 1)F\,[2\pi\varkappa\ (b-a)]^{-1/2}$ as a function of $b - a$, increasing monotonically.

3. Let the zone of separation be situated in the middle part of the stamp (Fig.1(5)). Then the boundary conditions of the problem will have the form (1.1) when $M = [a, b] \cup [c, d]$, $L = [b, c]$, $r_1 = r_2$. We must write

$$\begin{gathered} Z(z) = (z-a)^{-1/2+i\gamma}(z-b)^{-1/2-i\gamma}(z-c)^{-1/2+i\gamma}(z-d)^{-1/2-i\gamma} \\ Z(z) = z^{-2} + O(z^{-3}), \quad z \to \infty \end{gathered} \tag{3.1}$$

in the canonical solution (1.8).

Repeating the arguments of Sect.1, we obtain

$$\begin{gathered} \psi(z) = \frac{\pi}{2}(n^+ + n^- + 4) + \pi\alpha + \Psi(z) + \frac{n^+ - n^- - 2}{2i}\ln\frac{z-c}{z-b} \\ \Psi(z) = \gamma\ln\frac{(z-b)(z-c)}{(z-a)(z-d)} + \varphi(z) \end{gathered} \tag{3.2}$$

where $\varphi(z)$ from (1.17) is given in terms of elementary functions

$$\varphi(z) = 2\gamma\ln\left[\sqrt{\frac{z-a}{z-d}}\,\frac{\sqrt{(d-b)(z-c)} + \sqrt{(d-c)(z-b)}}{\sqrt{(b-a)(z-c)} + \sqrt{(c-a)(z-b)}}\right] \tag{3.3}$$

in the expansion at infinity (1.20)

$$\begin{gathered} \beta_0 = 2\gamma\ln\frac{\sqrt{d-b} + \sqrt{d-c}}{\sqrt{b-a} + \sqrt{c-a}} \\ \beta_1 = \gamma\left[\sqrt{(d-b)(d-c)} + \sqrt{(c-a)(b-a)}\right] \end{gathered} \tag{3.4}$$

Two linearly independent canonical solutions of the homogeneous problem (1.6), (1.7)

$$X_1(z) = -\frac{e^{i\varphi(z)}}{\sqrt{(z-a)(z-b)(z-c)(z-d)}}, \quad X_2(z) = \frac{ie^{i\varphi(z)}}{\sqrt{(z-a)(z-d)}} \tag{3.5}$$

are obtained from (1.8), (3.2) respectively when $n^+ = 0$, $n^- = -2$, $\alpha = 0$ and $n^+ = 0, n^- = -1$, $\alpha = -1$. Near the points $z = a$, $z = d$ the solutions oscillate and increase without limit. At the points $z = b$, $z = c$ the function $X_1(z)$ has a root-type singularity and the function $X_2(z)$ is bounded and non-zero.

The general solution of the problem is given by the formulas (1.24), (1.25), in which the constants C_1, C_2, D_1, D_2 satisfy, according to the condition at infinity (1.3), the equations

$$\begin{aligned} -C_1 + iD_1 &= [{}^1/_4\sigma_x^\infty + 2i\mu(\varkappa+1)^{-1}\varepsilon^\infty]\, e^{-i\beta_0} \\ -C_2 + iD_2 - C_1[i\beta_1 + {}^1/_2(a+b+c+d)] &+ \\ iD_1[i\beta_1 + {}^1/_2(a+d)] &= -{}^1/_2\pi^{-1}Fe^{i(\theta-\beta_0)} \end{aligned} \tag{3.6}$$

The constant C_3 is found from the given difference δ in tangential displacements between the edge points of the coupled segments $x = b$ and $x = c$, by the condition

$$\int_L u'(x)\,dx = u(c) - u(b) = \delta \tag{3.7}$$

Using the formulas (1.3), (1.24), (3.5), (3.6), (3.3) we obtain, when $x \in L$,

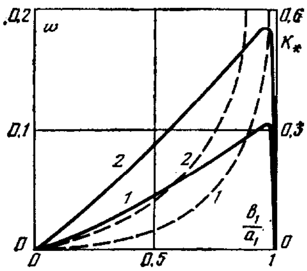

Fig.2

$$2\mu u'(x) = \frac{2\sqrt{\varkappa}\,P(x)\,\mathrm{ch}\,\rho(x)}{\sqrt{(x-a)(x-b)(c-x)(d-x)}} - \frac{2\sqrt{\varkappa}\,Q(x)\,\mathrm{sh}\,\rho(x)}{\sqrt{(x-a)(d-x)}} + \varkappa\Phi_0^-(x) + \Phi_0^+(x) \tag{3.8}$$

$$\rho(x) = \pi\gamma - \varphi_0(x)$$
$$\varphi_0(x) = 2\gamma\left[\mathrm{arctg}\sqrt{\frac{(d-c)(x-b)}{(d-b)(c-x)}} + \mathrm{arctg}\sqrt{\frac{(b-a)(c-x)}{(c-a)(x-b)}}\right] \tag{3.9}$$

where $\rho(x) \geqslant 0$, $\varphi_0(x) \geqslant 0$ on L.

The contact stresses on the slippage segment L are given by the formulas

$$\sigma_y(x) = -\frac{2P(x)\,\mathrm{sh}\,\varphi_0(x)}{\sqrt{(x-a)(x-b)(c-x)(d-x)}} - \frac{2Q(x)\,\mathrm{ch}\,\varphi_0(x)}{\sqrt{(x-a)(d-x)}} + \Phi_0^-(x) - \Phi_0^+(x) \tag{3.10}$$

As an example we shall consider in more detail the homogeneous problem in the case when the cut is symmetrical about the stamp edges, there is no preloading δ between the segments $[a, b]$ and $[c, d]$, and the stress and rotation at infinity are both zero.

Putting (3.3)-(3.5), (3.9) $a = -a_1$, $b = -b_1$, $c = b_1$, $d = a_1$ we obtain

$$\varphi(z) = 2\gamma\ln\left[\sqrt{\frac{z+a_1}{z-a_1}}\,\frac{\sqrt{(a_1+b_1)(z-b_1)} + \sqrt{(a_1-b_1)(z+b_1)}}{\sqrt{(a_1-b_1)(z-b_1)} + \sqrt{(a_1+b_1)(z-b_1)}}\right] \tag{3.11}$$

$$\beta_0 = 0, \quad \beta_1 = 2\gamma\sqrt{a_1^2 - b_1^2}, \quad \rho(x) = 2\gamma\,\mathrm{arctg}\sqrt{(b_1^2 - x^2)(a_1^2 - b_1^2)^{-1}}$$
$$\Phi_0(z) \equiv 0,\ C_1 = D_1 = 0,\ C_2 = {}^1/_2\pi^{-1}F\cos\theta,\ D_2 = {}^1/_2\pi^{-1}F\sin\theta \tag{3.12}$$

Substituting (3.12) into (3.7), (3.8) and taking into account the parity of the function $\rho(x)$, we obtain

$$C_3 = -\frac{1}{2\pi}Fb_1\omega\sin\theta, \quad \omega = \int_0^{b_1}\frac{\mathrm{sh}\,\rho(x)\,dx}{\sqrt{a_1^2 - x^2}}\left[b_1\int_0^{b_1}\frac{\mathrm{ch}\,\rho(x)\,dx}{\sqrt{(a_1^2 - x^2)(b_1^2 - x^2)}}\right]^{-1} \tag{3.13}$$

Here $\omega \in (0, 1)$, since both integrands are continuous, positive and the second function majorizes the first in $(0, b_1)$.

Figure 2 shows a graph of ω as a function of relative length of the slippage segment b_1/a_1 for $\varkappa = 1$ and $\varkappa = 3$ (the solid lines *1* and *2*).

Using (3.10), (3.12), (3.13) we obtain, on L,

$$\sigma_y(x) = \frac{F}{\pi}\left[\frac{(b_1\omega\sin\theta - x\cos\theta)\,\mathrm{sh}\,\varphi_0(x)}{\sqrt{(a_1^2 - x^2)(b_1^2 - x^2)}} + \frac{\sin\theta\,\mathrm{ch}\,\varphi_0(x)}{\sqrt{a_1^2 - x^2}}\right]$$
$$\varphi_0(x) = \pi\gamma - \rho(x) \tag{3.14}$$

Analysing the above formula we see that the contact stresses will be compressive over the whole slippage segment, provided that the following three conditions hold: $\sin\theta \leqslant 0$, $\omega\sin\theta \pm \cos\theta \leqslant 0$. This yields the following restriction on the direction of the force:

$$|\theta - {}^3/_2\pi| \leqslant \theta_0, \quad 0 \leqslant \theta_0 = \mathrm{arctg}\,\omega < {}^1/_4\pi$$

According to (1.2), (1.24), (3.5), (3.12)-(3.14), the asymptotic expansions for the stresses have the following form near the point $z = b_1$:

$$\sigma_y(x) = K_I[2\pi(x - b_1)]^{-1/2} + \sigma_0 + O(\sqrt{x - b_1}), \quad x \to b_1 + 0 \tag{3.15}$$

$$\sigma_y(x) = \sigma_0 + O(\sqrt{b_1 - x}), \quad x \to b_1 - 0$$
$$\tau_{xy}(x) = K_{II}[2\pi(b_1 - x)]^{-1/2} + O(\sqrt{b_1 - x}), \quad x \to b_1 - 0$$

$$K_I = -\frac{\varkappa - 1}{\varkappa + 1} K_{II}, \quad K_{II} = \frac{(\varkappa + 1) F b_1 (\cos\theta - \omega \sin\theta)}{2\sqrt{\pi \varkappa b_1 (a_1^2 - b_1^2)}} \tag{3.16}$$
$$\sigma_0 = \frac{(\varkappa + 1) F \sin\theta}{2\pi \sqrt{a_1^2 - b_1^2}}$$

Figure 2 shows how the dimensionless quantity $K_* = K_{II} F^{-1} \sqrt{2\pi a_1}$ varies as a function of b_1/a_1 in the case of a normal load ($\theta = {}^1/_2\pi$) for $\varkappa = 1,8$ and $\varkappa = 3$ (the dashed lines *1* and *2*). As expected, the tangential stress intensity coefficient K_{II} increases monotonically and without limit as the length of the separation segment increases. Thus, if the length of a closed crack reaches its critical value, it becomes globally unstable under a constant load and spreads over the whole segment $[-a_1, a_1]$.

REFERENCES

1. NAKHMEIN E.L. and NULLER B.M., On certain boundary value problems and their applications in the theory of elasticity. Izv. Vsesoyuz. n.-i. in-ta gidrotekhn., 172, 1984.
2. GALIN L.A., Impressing a stamp with friction and coupling. PMM, 19, 5, 1945.
3. FAL'KOVICH S.V., On a rigid stamp pressing on an elastic half-plane, in the presence of coupling and slippage segments. PMM, 19, 5, 1945.
4. DUNDURS G. and KOMNINOU M., Survey and future prospects of the study of the interphase crack. Mekhanika kompozit. materialov., 3, 1979.
5. ABRAMOV V.M., The prpblem of contact between an elastic half-plane and an absolutely rigid foundation, taking frictional forces into account. Dokl. AS SSSR, 17, 4, 1937.
6. MUSKHELISHVILI N.I., Some Basic Problems of the Mathematical Theory of Elasticity. Moscow, Nauka, 1966.
7. MUSKHELISHVILI N.I., Singular Integral Equations. Moscow, Nauka, 1980.
8. DUVAUT G. and LIONS J.L., Inequalities in Mechanics and Physics. Berlin, Springer Verlag, 1976.
9. CHEREPANOV G.P., Mechanics of Brittle Fracture. Moscow, Nauka, 1974.
10. SLEPYAN L.I., Mechanics of Cracks. Leningrad, Sudostroyeniye, 1981.

Translated by L.K.

PMM U.S.S.R.,Vol.50,No.4,pp.515-520,1986
Printed in Great Britain

0021-8928/86 $10.00+0.00

LYAPUNOV STABILITY AND SIGN DEFINITENESS OF A QUADRATIC FORM IN A CONE*

L.B. RAPOPORT

The use of the second Lyapunov method in many problems in the theory of stability of motion leads to the problem of sign definiteness of a quadratic form whose variables are defined in a convex polyhedral cone $C \subset R^n$. A method of obtaining the necessary and sufficient conditions is given for this problem. The conditions imposed on the elements of the third- and fourth-order matrices are given. The problem of asymptotic stability of a system with resonance /1/ is solved as an example.

A number of problems of the theory of the stability of motion require that the sign definiteness of the quadratic form be established, with conditions written in the form of linear inequalities. Usually, the conditions are those of non-negativity /1-3/, and the more general conditions can be reduced to them. The problem of sign definiteness of a quadratic form under the conditions of non-negativeness was considered for an arbitrary number of variables in /4/. However, the results obtained there can be reduced to the problem of the compatibility of systems of inequalities and pose well-known difficulties when used to solve specific problems.

The problem of the sign definiteness of a quadratic form in a convex cone

*Prikl.Matem.Mekhan.,50,4,674-679,1986